✓ 4.3.15 (b) - Thm 4.8

✓ 5.5.5 (a)

- 3.1.2

✓ 3.1.10 (a)(b)

✓ 3.1.11 (a)

- 3.1.17

✓ 3.2.8

✓ 3.2.21

✓ 3.4.6

✓ 5.2.5

- 5.3.12

4.3.15(b) Find the least square solution to $Ax = b$

where $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 5 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$

Want to minimize $\|Ax - b\|$.

Cols of A are linearly independent

$\Rightarrow$ Ker $A = \{0\}$

So, by Thm 4.8,

soln is $x^* = (A^T A)^{-1} A^T b$ ← use Matlab to calculate

Derivation of Thm 4.8:

$$Ax = b$$
$$\Rightarrow A^T (Ax) = A^T b$$
$$\Rightarrow (A^T A) x = A^T b$$
$$\Rightarrow x = (A^T A)^{-1} A^T b$$ ← $A^T A$ is inv. since Ker $A = \{0\}$

5.5.5(a) Find the orthogonal projection of $\begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}$ onto the span of $\underset{v_1}{\begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}}$.

First check if v_1, v_2 are orthogonal:

$$\left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = 2 - 1 - 1 = 0 \checkmark$$

orth. proj. of $\underset{v}{\begin{pmatrix}1\\2\\-1\\2\end{pmatrix}}$ onto $\operatorname{span}\left\{\underset{v_1}{\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}2\\1\\0\\-1\end{pmatrix}}\right\}$

$$w = \frac{\langle v, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle v, v_2\rangle}{\|v_2\|^2} v_2$$

$$= \frac{\left\langle \begin{pmatrix}1\\2\\-1\\2\end{pmatrix}, \begin{pmatrix}1\\-1\\2\\1\end{pmatrix}\right\rangle}{\left\langle \begin{pmatrix}1\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\\1\end{pmatrix}\right\rangle} \begin{pmatrix}1\\-1\\2\\1\end{pmatrix} + \frac{\left\langle \begin{pmatrix}1\\2\\-1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\0\\-1\end{pmatrix}\right\rangle}{\left\langle \begin{pmatrix}2\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\0\\-1\end{pmatrix}\right\rangle} \begin{pmatrix}2\\1\\0\\-1\end{pmatrix}$$

$$= \frac{-1}{7}\begin{pmatrix}1\\-1\\2\\1\end{pmatrix} + \frac{2}{6}\begin{pmatrix}2\\1\\0\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-1/7\\1/7\\-2/7\\-1/7\end{pmatrix} + \begin{pmatrix}2/3\\1/3\\0\\-1/3\end{pmatrix}$$

$$= \begin{pmatrix}11/21\\10/21\\-2/7\\-10/21\end{pmatrix}$$

3.1.10 Let V be an inner product space.

(a) Prove that $\langle x, v\rangle = 0$ for all $v \in V$ iff $x = 0$.

Proof: $\boxed{\Leftarrow}$ Assume $x = \vec{0}$, then $\langle x, x\rangle = 0$ ✓

$\boxed{\Rightarrow}$ Assume $\langle x, v\rangle = 0$ for all $v \in V$.

Then let $v = x$. Then $\langle x, x\rangle = 0$.

$\Rightarrow x = 0$ since $\langle v, v\rangle > 0$ whenever $v \neq 0$.

(b) Prove that $\langle x, v\rangle = \langle y, v\rangle$ for all $v \in V$ iff $x = y$.

Proof: $\boxed{\Leftarrow}$ Assume $x = y$, then $\langle x, v\rangle = \langle y, v\rangle \ \forall v \in V$

$\boxed{\Rightarrow}$ Assume $\langle x, v\rangle = \langle y, v\rangle$ for all $v \in V$.

Then $\langle x - y, v\rangle = \langle x, v\rangle - \langle y, v\rangle$

$= 0$ for all v,

By (a), this means that $x - y = 0$.

$\Rightarrow x = y$.

3.1.11 (a). Prove that $\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$

$$\|u+v\|^2 - \|u-v\|^2 = \langle u+v, u+v\rangle - \langle u-v, u-v\rangle$$

$$= \langle u, u\rangle + 2\langle u, v\rangle - \langle u, u\rangle + 2\langle u, v\rangle - \langle v, v\rangle$$

$$= 4\langle u, v\rangle$$

$\Rightarrow \langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$ ✓.

3.2.8 Prove that $\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta$, where θ is the angle bet $v + w$,

$$\|v-w\|^2 = \langle v-w, v-w\rangle$$
$$= \langle v,v\rangle - 2\langle v,w\rangle + \langle w,w\rangle$$
$$= \|v\|^2 + \|w\|^2 - 2\langle v,w\rangle$$

We know $\cos\theta = \dfrac{\langle v,w\rangle}{\|v\|\|w\|}$

$$\Rightarrow \langle v,w\rangle = \|v\|\|w\|\cos\theta$$

$$\Rightarrow \|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta \quad \checkmark .$$

(3.2.21) For what values of a, b are

$\begin{pmatrix}1\\1\\a\end{pmatrix}$ and $\begin{pmatrix}b\\-1\\1\end{pmatrix}$ orthogonal

(a) w/ respect to dot prod

(b) w/ respect to $\langle v,w\rangle = 3v_1w_1 + 2v_2w_2 + v_3w_3$

(a) $\begin{pmatrix}1\\1\\a\end{pmatrix}\cdot\begin{pmatrix}b\\-1\\1\end{pmatrix} = b - 1 + a = 0$

$$\Rightarrow a = 1 - b$$

(b) $\left\langle \begin{pmatrix}1\\1\\a\end{pmatrix}, \begin{pmatrix}b\\-1\\1\end{pmatrix}\right\rangle = 3b - 2 + a = 0$

$$\Rightarrow a = 2 - 3b .$$

3.4.6 Prove that if K is a pos. def. matrix, then for any positive scalar c, cK is also pos. def.

Proof :

$$(cK)^T = K^T c^T = \underbrace{K^T c}_{\text{since } c \text{ is a scalar}} = \underbrace{K c}_{\text{since } K \text{ is pos def.}} = cK$$

scalar matrix = matrix scalar

$$x^T (cK)x = c\, x^T K x > 0 \quad \text{since } c > 0 \text{ and } x^T K x > 0 \text{ for all } x \neq 0.$$

$\therefore$ cK is pos. def.

5.2.5 Find an orthogonal basis for

$$\text{span}\left\{ \underset{w_1}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{pmatrix}}, \underset{w_3}{\begin{pmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{pmatrix}} \right\}$$

$v_1 = w_1$.

$$v_2 = w_2 - \frac{\langle v_1, w_2 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_3 - \frac{\langle v_1, w_3 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_2, w_3 \rangle}{\|v_2\|^2} v_2$$

Ans $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.